

# PERSISTENCE OF NONCOMPACT NORMALLY HYPERBOLIC INVARIANT MANIFOLDS IN BOUNDED GEOMETRY

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**ABSTRACT.** We prove a persistence result for noncompact normally hyperbolic invariant manifolds in Riemannian manifolds of bounded geometry. The bounded geometry of the ambient manifold is a crucial assumption in order to control the uniformity of all estimates throughout the proof.

## 1. INTRODUCTION

Normally hyperbolic invariant manifolds (NHIMs for short) are used in many areas of dynamical systems, for example, in singular perturbation theory. It is well-known that compact NHIMs are persistent under any  $C^1$ -small perturbation, see [4, 6], while Sakamoto [7] and Bates, Lu, and Zeng [1] have extended this to noncompact NHIMs in Euclidean and Banach spaces respectively. Our result is an extension to a general noncompact setting in Riemannian manifolds. Bounded geometry is a crucial additional ingredient, needed to formulate the necessary uniformity conditions which allow to replace compactness by uniformity throughout the proof. Bounded geometry can be viewed as a uniformity condition on the ambient manifold and is automatically satisfied for Euclidean space.

## 2. BOUNDED GEOMETRY

We follow Eichhorn [2] to define bounded geometry. Recall that the injectivity radius  $r_{\text{inj}}(x)$  at a point  $x \in Q$  is the maximum radius for which the exponential map at  $x$  is a diffeomorphism, and that normal coordinates are defined as the inverse map.

**Definition 2.1.** *We say that a complete, finite-dimensional Riemannian manifold  $(Q, g)$  has  $k$ -th order bounded geometry when*

- (1) *the global injectivity radius  $r_{\text{inj}}(Q) = \inf_{x \in Q} r_{\text{inj}}(x)$  is positive,  $r_{\text{inj}}(Q) > 0$ ;*
- (2) *the Riemannian curvature  $R$  and its covariant derivatives up to  $k$ -th order are uniformly bounded,*

$$\forall 0 \leq i \leq k: \sup_{x \in Q} \|\nabla^i R(x)\| < \infty,$$

*with operator norm of  $\nabla^i R(x)$  viewed as a multilinear map on  $T_x Q$ .*

Note that both Euclidean space and compact smooth Riemannian manifolds have bounded geometry of any order  $k$ , i.e.  $k = \infty$ . Less trivial examples of bounded geometry are symmetric spaces or spaces constructed as products or as compactly glued connected sums of bounded geometry spaces.

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It follows from Theorem 2.4 in [2] that a manifold of bounded geometry has an atlas of normal coordinate charts such that for some fixed  $\delta > 0$  there is a normal coordinate chart defined on each ball  $B(x; \delta)$ , and moreover, the representation of the metric  $g$  in each chart is  $C^k$ -bounded, uniformly over all charts. We shall work with this preferred atlas and measure the  $C^k$  norm of functions in the following way.

**Definition 2.2.** *Let  $X, Y$  be Riemannian manifolds of  $k+1$ -bounded geometry and  $f \in C^k(X; Y)$ . We say that  $f$  is of class  $C_b^k$  when there exist  $\delta_X, \delta_Y > 0$  such that for each  $x \in X$  we have  $f(B(x; \delta_X)) \subset B(f(x); \delta_Y)$  and the representation*

$$\tilde{f}_x = \exp_{f(x)}^{-1} \circ f \circ \exp_x : B(0; \delta_X) \subset T_x X \rightarrow T_y Y \quad (1)$$

*in normal coordinates is of class  $C_b^k$  (i.e.  $C^k$ -bounded), and moreover, the associated  $C^k$ -norms of  $\tilde{f}_x$  are bounded uniformly in  $x \in X$ .*

This is a natural definition:  $k+1$ -bounded geometry implies that coordinate transition maps are uniformly  $C^k$ -bounded, hence this definition is equivalent to measuring the  $C^k$ -norm of  $f$  at  $x$  in any normal coordinate chart  $B(x'; \delta_X)$  containing  $x$ . Classes  $C_{b,u}^k(X; Y)$  and  $C_{b,u}^{k,\alpha}(X; Y)$  of uniformly (Hölder) continuous functions can be defined analogously when  $X, Y$  are of  $k+2$ -bounded geometry. These ideas can also be extended to classes  $C_b^{k,\alpha}$  and  $C_{b,u}^{k,\alpha}$  of vector fields and submanifolds. We shall allow submanifolds to be non-injectively immersed.

### 3. RESULTS

We use the following definition of normal hyperbolicity. The flow is assumed complete for simplicity.

**Definition 3.1.** *Let  $(Q, g)$  be a smooth Riemannian manifold,  $\Phi^t \in C^{r \geq 1}$  a flow on  $Q$ , and let  $M \in C^{r \geq 1}$  be a submanifold of  $Q$ . Then  $M$  is called a normally hyperbolic invariant manifold of the dynamical system  $(\mathbb{R}, Q, \Phi)$  if all of the following conditions hold true:*

- (1)  *$M$  is invariant, i.e.  $\forall t \in \mathbb{R}: \Phi^t(M) = M$ ;*
- (2) *there exists a continuous splitting*

$$T_M Q = TM \oplus E^+ \oplus E^- \quad (2)$$

*of the tangent bundle  $TQ$  over  $M$  with globally bounded, continuous projections  $\pi_M, \pi_+, \pi_-$  and this splitting is invariant under the linearized flow  $D\Phi^t = D\Phi_M^t \oplus D\Phi_+^t \oplus D\Phi_-^t$ ;*

- (3) *there exist real numbers  $\rho_- < -\rho_M \leq 0 \leq \rho_M < \rho_+$  and  $C_M, C_+, C_- > 0$  such that the following exponential growth conditions hold on the various subbundles:*

$$\begin{aligned} \forall t \in \mathbb{R}, (m, x) \in TM: \quad & \|D\Phi_M^t(m) x\| \leq C_M e^{\rho_M |t|} \|x\|, \\ \forall t \leq 0, (m, x) \in E^+ : \quad & \|D\Phi_+^t(m) x\| \leq C_+ e^{\rho_+ t} \|x\|, \\ \forall t \geq 0, (m, x) \in E^- : \quad & \|D\Phi_-^t(m) x\| \leq C_- e^{\rho_- t} \|x\|. \end{aligned} \quad (3)$$

This definition corresponds to ‘eventual absolute normal hyperbolicity’ in [6], and is slightly more restrictive than the ‘relative normal hyperbolicity’ definition in [6] that is also used in [4].

We say that  $M$  is an  $r$ -NHIM if the more general spectral gap condition

$$\rho_- < -r \rho_M \leq 0 \leq r \rho_M < \rho_+ \quad \text{with } r \geq 1 \quad (4)$$

on the growth exponents above is satisfied.

**Theorem 3.2.** *Let  $k \geq 2$ ,  $\alpha \in [0, 1]$  and  $r = k + \alpha$ . Let  $(Q, g)$  be a smooth Riemannian manifold of bounded geometry and  $v \in C_{b,u}^{k,\alpha}$  a vector field on  $Q$ . Let  $M \in C_{b,u}^{k,\alpha}$  be a connected, complete submanifold of  $Q$  that is  $r$ -normally hyperbolic for the flow defined by  $v$ , with empty unstable bundle, i.e.  $\text{rank}(E^+) = 0$ .*

*Then for each sufficiently small  $\eta > 0$  there exists a  $\delta > 0$  such that for any vector field  $\tilde{v} \in C_{b,u}^{k,\alpha}$  with  $\|\tilde{v} - v\|_1 < \delta$ , there is a unique submanifold  $\tilde{M}$  in the  $\eta$ -neighborhood of  $M$ , such that  $\tilde{M}$  is diffeomorphic to  $M$  and invariant under the flow defined by  $\tilde{v}$ . Moreover,  $\tilde{M}$  is  $C_{b,u}^{k,\alpha}$  and the distance between  $\tilde{M}$  and  $M$  can be made arbitrarily small in  $C^{k-1}$ -norm by choosing  $\|\tilde{v} - v\|_{k-1}$  sufficiently small.*

Let us make some remarks on this result.

- (1) The spectral gap condition (4) of  $r$ -normal hyperbolicity is essential to the proof. The  $C^{k,\alpha}$  smoothness result is optimal. The minimum smoothness requirement  $k \geq 2$  is a stronger assumption than  $k \geq 1$  in the well-known compact case. This seems to be intrinsic to the noncompact case, cf. hypothesis H2 in [1]. If the spectral gap condition only holds for some  $1 \leq r < 2$ , then we can still obtain a perturbed manifold  $\tilde{M}$ , but this manifold will generally not have better than  $C^r$  smoothness.
- (2) It should be possible to improve this result by lifting some of the technical restrictions. First of all, an unstable bundle  $E^+$  can be added for full normal hyperbolicity. It should hold that the persistent manifold  $\tilde{M}$  is an  $r$ -NHIM again.
- (3) Definition 3.1 could be relaxed to the more general definition of ‘relative normal hyperbolicity’ as used in [4, 6, 1]. This would require using the graph transform method; our Perron method proof seems tied to the current definition.
- (4) We only obtain a  $C^{k-1}$ -norm estimate for the perturbation distance of  $\tilde{M}$  away from  $M$ , even though  $\tilde{M} \in C^{k,\alpha}$  is preserved. It should be possible to improve this to the perturbation being  $C^{k,\alpha}$ -small when  $\|\tilde{v} - v\|_{k,\alpha}$  is small.

By standard phase space extension techniques, we obtain the following results as a corollary.

**Corollary 3.3.** *Assume the setting of Theorem 3.2. If the vector field  $\tilde{v}$  also depends on time, i.e.  $\tilde{v} \in C_{b,u}^{k,\alpha}(\mathbb{R} \times Q)$ , then there still exists a persistent manifold  $\tilde{M} \in C_{b,u}^{k,\alpha}$ , although it may be time-dependent. Similarly, if the vector field  $\tilde{v}$  depends on an external parameter  $p \in \mathbb{R}^n$  and  $M$  is an  $r$ -NHIM for  $p = 0$ , then there exists a neighborhood  $U \ni 0$  such that for each  $p \in U$  we have a unique persistent manifold  $\tilde{M}_p \in C_{b,u}^{k,\alpha}$  and  $\tilde{M}_p$  depends  $C^{k,\alpha}$  on  $p$ .*

#### 4. IDEA OF THE PROOF OF THEOREM 3.2

The following is only a rough sketch of the proof of Theorem 3.2, for a detailed exposition see [3].

We first reduce the problem to a trivial bundle  $X \times Y$  where  $X$  is constructed as a manifold of bounded geometry of sufficiently high order (say  $k + 10$ ) that approximates  $M$ . Then we embed the normal bundle  $N$  of  $X$  into  $X \times Y$  with  $Y = \mathbb{R}^n$  for some  $n$ . A uniform tubular neighborhood of  $M$  can be modeled on  $N$  since  $M$  is the graph of a small function  $h: X \rightarrow Y$ .

Additional normally hyperbolic dynamics can be added in the directions of  $Y$  complementary to  $N$ .

We apply a generalization of the Perron method based on ideas in [5]. Let  $\tilde{v}_x$  and  $\tilde{v}_y(x, y) = A(x)y + f(x, y)$  denote the horizontal and vertical parts of the vector field  $\tilde{v}$  respectively. If  $(x(t), y(t))$  is a curve in  $X \times Y$  with  $y(t)$  uniformly small, then we denote by  $\Phi_y(t, t_0, x_0)$  the flow of  $\tilde{v}_x(\cdot, y(t))$  and by  $\Psi_x(t, t_0)$  the linear flow of  $A(x(t))$  on  $Y$ . A contraction map  $T$  is defined by  $T(y, x_0) = T_Y(T_X(y, x_0), y)$  with

$$\begin{aligned} T_X(y, x_0)(t) &= \Phi_y(t, 0, x_0), \\ T_Y(x, y)(t) &= \int_{-\infty}^t \Psi_x(t, \tau) f(x(\tau), y(\tau)) \, d\tau \end{aligned} \tag{5}$$

mappings into appropriate spaces of curves in  $X, Y$  respectively. We finally recover the persistent manifold  $\tilde{M}$  as the graph of the map  $\tilde{h}: x_0 \mapsto \Theta(x_0)(0)$  where  $\Theta$  denotes the fixed point of  $T$  as a function of the parameter  $x_0 \in X$ ; the curve  $\Theta(x_0)$  in  $Y$  is then evaluated at  $t = 0$ .

The  $C_{b,u}^{k,\alpha}$  smoothness of  $\Theta$  is proven inductively using ideas in [8] and the fiber contraction theorem. We introduce certain formal tangent bundles to work around the problem that the spaces of curves in  $X, Y$  are not (Banach) manifolds. We relate holonomy along closed loops in  $X$  to the curvature (which is bounded) to prove uniform continuity of the formal derivatives of  $T$ . Restricting the spaces of curves in  $X, Y$  to bounded time intervals turns these into Banach manifolds; this we use to finally recover true derivatives that lead to  $\tilde{M} \in C_{b,u}^{k,\alpha}$ .

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